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A Corrected Proof of a Result of Grinold

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In this note we correct the proof of a result of Grinold [4] on continuous linear programs. First we define the continuous linear programming problem as follows:

$$\text{CLP: maximize} \qquad \int_0^T a(t)\, z(t)\, dt$$

subject to

$$B(t)\, z(t) \leqslant c(t) + \int_0^t K(t, s)\, z(s)\, ds, \qquad z(t) \geqslant 0, \quad t \in [0, T].$$

For each $t \in [0, T]$, $B(t)$ is an $m \times n$ matrix, $c(t) \in R^m$, $a(t)$ and $z(t) \in R^n$ and, for each $s \leqslant t$, $K(t, s)$ is an $m \times n$ matrix. The components of B, K, a and c are all bounded measurable functions and z is to be chosen from $L_1^n[0, T]$.

We write $P(c)$ for the set of feasible z for CLP, taken to depend on c. If y is an m-vector and A is an $m \times n$ matrix we let

$$L(A, y) = \{x \in R^n : Ax \leqslant y, x \geqslant 0\}.$$

Write $\bar{L}(A, y)$ for the convex hull of the extreme points of $L(A, y)$.

The notation here is taken for the most part from the paper of Grinold [4]. Following Grinold we introduce the following conditions on CLP:

(A) (Algebraic condition): For each $t \in [0, T]$, $B(t)z \leqslant 0$, $z \geqslant 0$ implies $K(t, s)z \leqslant 0$ for all $s \leqslant t$.

(B) (Boundedness condition): There is a $\rho > 0$ such that for each $t \in [0, T]$ and $d \in R^m$, $x \in \bar{L}(B(t), d)$ implies $\|x\| \leqslant \rho \|d\|$.

It is easy to see that the recession cone for $L(A, y)$ consists of points $x \in R^n$ with $x \geqslant 0$ and $Ax \leqslant 0$. Thus from a well-known result on recession cones (see, for example, Holmes [7]) for each x in $L(A, y)$, $x = v_1 + v_2$ with $v_1 \in \bar{L}(A, y)$ and $v_2 \geqslant 0$, $Av_2 \leqslant 0$. Grinold [3] has shown that if $x(t)$, $y(t)$ are measurable with $x(t) \in L(A, y(t))$, $t \in [0, T]$, then $x(t) = v_1(t) + v_2(t)$ with

$$v_1(t) \in \bar{L}(A, y(t)), \qquad v_2(t) \geqslant 0, \qquad Av_2(t) \leqslant 0$$

and both v_1 and v_2 measurable.

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Now we state the result of this paper, which was first given by Grinold [4].

THEOREM. *If conditions* (A) *and* (B) *hold and* $z \in P(c)$ *then* $z = v + w$ *with*

(i) $v \in P(c)$, $\|v(t)\| \leqslant \delta \|c\|$ *for almost all* $t \in [0, T]$, *where* δ *is independent of* c *and* $\|c\|$ *is the* L_∞ *norm of* c,

(ii) $B(t)\, w(t) \leqslant 0$, $w(t) \geqslant 0$.

Before giving a new proof of this result we observe that this theorem is central to Grinold's papers [4, 5] upon which the results of Farr and Hanson [1, 2] are dependent. The result has also been used by Levine and Pomerol [8]. A proof of a similar result for the discrete case can be found in Grinold [6]. However, the proof of the theorem given by Grinold [4] contains an error and the v and w that he constructs will not necessarily have the required properties. To demonstrate this consider the following example.

Take $n = m = 1$, $B(t) = -1$, $K(t, s) = -1$ and $c(t)$ a constant value γ. Then if γ is large and negative $z(t) = \gamma^2$ will be feasible for CLP and it is easy to see that conditions (A) and (B) hold. Now $\bar{L}(B(t), d) = \{\max(0, -d)\}$ and using the construction that Grinold suggests gives $v(t) = t\gamma^2 - \gamma$, which is not bounded by any multiple of $|\gamma|$.

Proof. Suppose that $z(t)$, $t \in [0, T]$ is feasible for CLP. Let $d(t) = c(t) + \int_0^t K(t, s)\, z(s)\, ds$. Thus $z(t) \in L(B, d(t))$. Then from the remarks above we can find v_1, w_1 measurable with $z(t) = v_1(t) + w_1(t)$, where

$$v_1(t) \in \bar{L}(B(t), d(t)), \qquad w_1(t) \geqslant 0, \qquad B(t)\, w_1(t) \leqslant 0.$$

From condition (A) we have $K(t, s)\, w_1(t) \leqslant 0$. Thus if we define $d_1(t)$ by

$$d_1(t) = c(t) + \int_0^t K(t, s)\, v_1(s)\, ds,$$

we have $d(t) \leqslant d_1(t)$ and so $v_1(t) \in L(B(t), d_1(t))$.

Now we can repeat this process obtaining a sequence of measurable functions v_n, d_n with

$$v_{n-1}(t) = v_n(t) + w_n(t),$$

$$v_n(t) \in \bar{L}(B(t), d_{n-1}(t)), \qquad w_n \geqslant 0, \quad B(t)\, w_n(t) \leqslant 0,$$

$$d_n(t) = c(t) + \int_0^t K(t, s)\, v_n(s)\, ds,$$

and, as $d_{n-1}(t) \leqslant d_n(t)$,

$$v_n(t) \in L(B(t), d_n(t)).$$

Since $v_n(t)$ is monotonic decreasing for each t, it has a limit and we define the measurable function v by

$$v(t) = \lim_{n \to \infty} v_n(t).$$

Moreover $0 \leqslant v_n(t) \leqslant z(t)$ and the integral

$$\int_0^t K(t, s)\, z(s)\, ds$$

is finite. So by Lebesgue's bounded convergence theorem $K(t, s)\, v(s)$ is integrable and

$$\lim_{n \to \infty} d_n(t) = c(t) + \int_0^t K(t, s)\, v(s)\, ds.$$

We write $d'(t)$ for this function. The extreme points of $L(B(t), d_n(t))$ approach those of $L(B(t), d'(t))$ and this is sufficient to show that $v(t) \in \bar{L}(B(t), d'(t))$.

Now from condition (B)

$$\begin{aligned} \|v(t)\| &\leqslant \rho\, \|d'(t)\| \\ &\leqslant \rho\, \|c(t)\| + \rho \left\| \int_0^t K(t, s)\, v(s)\, ds \right\| \\ &\leqslant \rho\, \|c\| + \rho\alpha \int_0^t \|v(s)\|\, ds, \qquad t \in [0, T], \end{aligned}$$

where $\|c\|$ is the L_∞ norm of c and α is the essential supremum of $\|K(t, s)\|$. Hence using Gronwall's lemma [9, p. 75],

$$\|v(t)\| \leqslant \rho\, \|c\|\, e^{\rho\alpha t}, \qquad \text{for almost all } t \in [0, T],$$

and we set $\delta = \rho e^{\rho\alpha T}$.

Now define w by $w(t) = z(t) - v(t)$. Then

$$w(t) = \lim_{n \to \infty} \sum_{i=1}^{n} w_i(t),$$

and so $B(t)\, w(t) \leqslant 0$ and $w(t) \geqslant 0$, $t \in [0, T]$. This completes the proof of the theorem.

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